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LETTER TO THE EDITOR

Operator content of the three-state Potts quantum chain

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Abstract. We compute the lowest excitations of the spectra of the three-state Potts quantum chain with periodic and twisted boundary conditions. These spectra can be understood in terms of eighteen irreducible representations of the Virasoro algebra with $c = \frac{4}{5}$.

In this letter we consider the three-state Potts quantum chain defined by the Hamiltonian

$$H = -\frac{2}{3\sqrt{3}} \sum_{i=1}^{N} [\sigma_i + \sigma_i^+ + \lambda(\Gamma_i \Gamma_{i+1}^+ + \Gamma_i^+ \Gamma_{i+1})] \tag{1}$$

where λ is the inverse of the temperature, N represents the number of sites, σ and Γ are the matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

and $\omega = \exp(\frac{2}{3}\pi i)$. This Hamiltonian is self-dual and has a critical point at $\lambda = 1$. Our aim is to find the whole spectrum at the critical point in the finite-size scaling limit and express it in terms of irreducible representations (irreps) of the Virasoro algebra (Cardy 1986). A preliminary step in this direction was described in a previous publication (von Gehlen *et al* 1985).

For periodic as well as twisted boundary conditions we have calculated the 6–8 lowest states of each sector for always 3–13 sites using the Lanczos (1950) method. For zero momentum and periodic boundary conditions we went up to 14 sites. The results for up to 9 sites were checked by exact diagonalisation.

We specify the boundary conditions taking in (1):

$$\Gamma_{N+1} = \omega^{\tilde{Q}} \Gamma_1 \qquad (\tilde{Q} = 0, 1, 2) \tag{3}$$

and obtain the Hamiltonians $H^{(\tilde{Q})}$. If we take $\Gamma_{N+1} = 0$ (free boundary conditions) we have $H^{(F)}$. The overall factor $2/(3\sqrt{3})$ in the Hamiltonian (1) (which fixes the Euclidean time scale in a conformal theory) is taken from von Gehlen *et al* (1986). That this is the proper factor can be checked by looking at the quadratic finite-size corrections to the ground-state energy per site for periodic boundary conditions:

$$-\frac{E_0^{(P)}}{N} = a_0 + \frac{\pi}{6} \frac{a_2^{(P)}}{N^2} + \dots \tag{4}$$

and for free boundary conditions:

$$-\frac{E_0^{(F)}}{N}=a_0+\frac{a_1^{(F)}}{N}+\frac{\pi}{24}\frac{a_2^{(F)}}{N^2}+\ldots. \tag{5}$$

From conformal invariance we have (Blöte *et al* 1986)

$$a_2^{(P)}=c \tag{6}$$

and (von Gehlen and Rittenberg 1986)

$$a_2^{(F)}=c. \tag{7}$$

In (6) and (7), c is the central charge of the Virasoro algebra equal to $\frac{4}{5}$ for the three-state Potts model (Friedan *et al* 1984, Dotsenko 1984). Since a_0 is known exactly (Hamer 1981)

$$a_0=\frac{8}{9\sqrt{3}}+\frac{4}{3\pi} \tag{8}$$

we fit $a_2^{(P)}$, $a_1^{(F)}$ and $a_2^{(F)}$ to the numerical results and obtain

$$a_2^{(P)}=0.800\,08(1) \qquad a_1^{(F)}=-0.230\,193(1) \qquad a_2^{(F)}=0.792(1) \tag{9}$$

in agreement with $c=\frac{4}{5}$.

Since the Hamiltonian (1) is Z_3 invariant, each of the matrices $H^{(\tilde{Q})}$ has a block-diagonal form $H_Q^{(\tilde{Q})}$ corresponding to the charge sector $Q=0, 1$ and 2 of $H^{(\tilde{Q})}$. At $\lambda=1$ self-duality and the supplementary Z_2 symmetry of the Hamiltonian (1) give the following relations among the matrices $H_Q^{(\tilde{Q})}$:

$$H_Q^{(\tilde{Q})}=H_{\tilde{Q}}^{(Q)} \qquad H_1^{(\tilde{Q})}=H_2^{(\tilde{Q})}. \tag{10}$$

(The equalities among the matrices imply only that the spectra are the same.) We are thus left with only three independent matrices $H_0^{(0)}$, $H_1^{(0)}$ and $H_1^{(1)}$. We can further prediagonalise these matrices using their translational invariance and we will denote by $E_0^{(0)}(P)$, $E_1^{(0)}(P)$ and $E_1^{(1)}(P)$ the eigenvalues corresponding to the momenta P. ($E_0^{(P)}$ is the lowest eigenvalue of $E_0^{(0)}(0)$.) These eigenvalues depend on the number of sites N of the chain.

We now consider the following quantities which are relevant for finite-size scaling:

$$\mathscr{E}_0^{(0)}(P)=\lim_{N\to\infty}\frac{N}{2\pi}(E_0^{(0)}(P)-E_0^{(P)}) \qquad \mathscr{E}_1^{(0)}(P)=\lim_{N\to\infty}\frac{N}{2\pi}(E_1^{(0)}(P)-E_0^{(P)})$$

$$\mathscr{E}_1^{(1)}(P)=\lim_{N\to\infty}\frac{N}{2\pi}(E_1^{(1)}(P)-E_0^{(P)}). \tag{11}$$

These quantities have been estimated numerically using Van den Broeck–Schwartz (1979) approximants and are given in tables 1, 2 and 3 for momenta up to three. For higher levels of certain sectors one encounters the problem that the sequences in N cross over and so in general we have been able to determine the $\mathscr{E}(P)$ up to the fifth or sixth level only. In this context notice the approximate degeneracy of the approximants corresponding to some levels. Altogether, we have determined 85 levels.

This concludes the experimental part of our 'spectroscopic' work. We now turn to the 'theoretical' part of this letter where we try to explain the 'observed' spectra.

Table 1. The spectrum $\mathscr{E}_0^{(0)}(P)$. The Van den Broeck-Schwartz approximants for the levels are denoted by $\mathscr{E}_0^{(0)}(P)$ ('experimental'). The contributions of each irrep $(\Delta, \bar{\Delta})$ to the spectrum according to (13) is shown. The numbers under each $(\Delta, \bar{\Delta})$ indicate the degeneracy. The levels marked by an asterisk are doubly degenerate (parity doublets) even for finite chains. The spectrum for negative momenta is the same as for positive momenta. The figures in brackets in the last column indicate the estimated error.

P	$\Delta+r+\bar{\Delta}+\bar{r}$	$(0,0)$	$(\frac{2}{5},\frac{2}{5})$	$(\frac{7}{5},\frac{2}{5})$	$(\frac{2}{5},\frac{7}{5})$	$(\frac{7}{5},\frac{7}{5})$	$(3,0)$	$\mathscr{E}_0^{(0)}(P)$ ('experimental')
0	0.8	—	1	—	—	—	—	0.820 (3)
	2.8	—	1	1	1	1	—	2.79 (1)*; 2.817 (2); 2.832 (2)
	4.0	1	—	—	—	—	—	3.996 (4)
	4.8	—	1	1	1	1	—	4.77 (2)*; 4.82 (2); 4.83 (1)
1	1.8	—	1	1	—	—	—	1.798 (3); 1.824 (4)
	3.8	—	1	1	1	1	—	3.78 (2); 3.78 (1); 3.82 (1); 3.83 (2)
2	2	1	—	—	—	—	—	1.999 98 (4)
	2.8	—	1	1	—	—	—	2.77 (8); 2.8 (1)
	4.8	—	2	1	2	1	—	4.75 (6); 4.77 (5); 4.82 (4)
3	3	1	—	—	—	—	1	2.995 (5); 2.999 (1)
	3.8	—	2	1	—	—	—	

Table 2. The spectrum $\mathscr{E}_1^{(0)}(P)$. Double degeneracy marked by * as in table 1.

P	$\Delta+r+\bar{\Delta}+\bar{r}$	$(\frac{1}{15},\frac{1}{15})$	$(\frac{2}{3},\frac{2}{3})$	$\mathscr{E}_1^{(0)}(P)$ ('experimental')
0	$\frac{2}{25}\approx 0.133$	1	—	0.1333 (1)
	$\frac{4}{3}\approx 1.333$	—	1	1.3333 (5)
	$\frac{32}{15}\approx 2.133$	1	—	2.139 (1)
	$\frac{10}{3}\approx 3.333$	—	1	3.333 (4)
	$\frac{62}{15}\approx 4.133$	4	—	4.13 (2)*; 4.138 (5); 4.18 (1)
	$\frac{16}{3}\approx 5.333$	—	4	5.31 (7); 5.333 (1)*; 5.334 (5)
1	$\frac{17}{15}\approx 1.133$	1	—	1.1344 (5)
	$\frac{7}{3}\approx 2.333$	—	1	2.332 (5)
	$\frac{47}{15}\approx 3.133$	2	—	3.10 (5); 3.13 (5)
	$\frac{13}{3}\approx 4.333$	—	2	4.329 (5); 4.332 (3)
2	$\frac{32}{15}\approx 2.133$	2	—	2.132 (1); 2.134 (3)
	$\frac{10}{3}\approx 3.333$	—	2	3.32 (2); 3.332 (6)
	$\frac{62}{15}\approx 4.133$	3	—	4.12 (3); 4.124 (5)
3	$\frac{47}{15}\approx 3.133$	3	—	3.13 (2); 3.13 (1); 3.13 (3)
	$\frac{13}{3}\approx 4.333$	—	2	4.3 (1); 4.33 (2)
	$\frac{77}{15}\approx 5.133$	5	—	5.15 (5)

Let Δ be the lowest weight of an irreducible representation (irrep) of the Virasoro algebra with $c=\frac{4}{5}$. Its possible values are (Friedan *et al* 1984)

$$\Delta=\Delta(p,q)=\Delta(5-p,6-q)=[(5q-6p)^2-1]/120 \qquad (1\leqslant p\leqslant 4, 1\leqslant q\leqslant 5). \tag{12}$$

The degeneracy of the level $(\Delta+r)$ $(r=0,1,2,\ldots)$ will be denoted by $d(\Delta, r)$. Using the character formula of Rocha-Caridi (1984), Altschüler and Lacki (1985) have computed for us the values of $d(\Delta, r)$, which are shown in table 4. Keeping in mind that for a quantum chain with periodic boundary conditions $(\tilde{Q}=0)$ and twisted boundary conditions $(\tilde{Q}=1)$, the spectra are given by two Virasoro algebras with the same central charge, $\mathscr{E}_0^{(0)}(P)$, $\mathscr{E}_1^{(0)}(P)$ and $\mathscr{E}_1^{(1)}(P)$ receive contributions from a pair of

Table 3. The spectrum $\mathscr{E}_1^{(1)}(P)$. In this case the spectrum for negative momenta is different than for positive momenta (equation (13*b*)).

P	$\Delta+r+\bar\Delta+\bar r$	$(\frac{2}{5},\frac{1}{15})$	$(0,\frac{2}{3})$	$(\frac{7}{5},\frac{1}{15})$	$(3,\frac{2}{3})$	$\mathscr{E}_1^{(1)}(P)$ ('experimental')
-2	$\frac{5}{3}\approx 1.666$	—	1	—	—	1.656 (5)
	$\frac{37}{15}\approx 2.466$	2	—	—	—	2.39 (1); 2.5 (1)
	$\frac{67}{15}\approx 4.466$	3	—	3	—	4.28 (2); 4.4 (2); 4.4 (1)
-1	$\frac{2}{3}\approx 0.666$	—	1	—	—	0.666 66 (3)
	$\frac{22}{15}\approx 1.466$	1	—	—	—	1.47 (2)
	$\frac{52}{15}\approx 3.466$	2	—	2	—	3.45 (2); 3.47 (2); 3.48 (1); 3.51 (3)
	$\frac{14}{3}\approx 4.666$	—	2	—	—	
0	$\frac{7}{15}\approx 0.466$	1	—	—	—	0.4667 (3)
	$\frac{37}{15}\approx 2.466$	1	—	1	—	2.460 (5); 2.478 (2)
	$\frac{11}{3}\approx 3.666$	—	1	—	—	3.665 (8)
	$\frac{67}{15}\approx 4.466$	2	—	2	—	4.45 (2); 4.44 (1)
1	$\frac{22}{15}\approx 1.466$	1	—	1	—	1.466 (5); 1.469 (2)
	$\frac{8}{3}\approx 2.666$	—	1	—	—	2.667 (2)
	$\frac{52}{15}\approx 3.466$	1	—	1	—	3.45 (2); 3.47 (2)
	$\frac{14}{3}\approx 4.666$	—	1	—	1	4.67 (5)
2	$\frac{37}{15}\approx 2.466$	1	—	1	—	2.43 (3); 2.44 (6)
	$\frac{11}{3}\approx 3.666$	—	1	—	1	3.65 (2); 3.663 (3)
	$\frac{67}{15}\approx 4.466$	2	—	2	—	4.4 (1); 4.46 (4); 4.47 (5)

Table 4. The function $d(\Delta, r)$ representing the degeneracy of the level $(\Delta+r)$ of the irreducible representation with lowest weight Δ.

			r										
p	q	Δ	0	1	2	3	4	5	6	7	8	9	10
1	1	0	1	0	1	1	2	2	4	4	7	8	12
2	1	$\frac{2}{5}$	1	1	1	2	3	4	6	8	11	15	20
2	2	$\frac{1}{40}$	1	1	2	3	4	6	9	12	17	23	31
3	1	$\frac{7}{5}$	1	1	2	2	4	5	8	10	15	19	26
3	2	$\frac{21}{40}$	1	1	2	3	5	7	10	14	19	26	35
3	3	$\frac{1}{15}$	1	1	2	3	5	7	10	14	20	26	36
4	1	3	1	1	2	3	4	5	8	10	14	18	24
4	2	$\frac{13}{8}$	1	1	2	3	4	6	9	12	16	22	29
4	3	$\frac{2}{3}$	1	1	2	2	4	5	8	10	15	19	27
4	4	$\frac{1}{8}$	1	1	1	2	3	4	6	8	11	15	20

irreps. We denote them by Δ and $\bar\Delta$ where $x=\Delta+\bar\Delta$ and $s=\Delta-\bar\Delta$ are the scaling dimensions and the spin corresponding to the primary field associated with $(\Delta,\bar\Delta)$. If an irrep $(\Delta,\bar\Delta)$ contributes to the spectrum $\mathscr{E}(P)$ one obtains the levels

$$\mathscr{E}(P)=(\Delta+r)+(\bar\Delta+\bar r) \tag{13a}$$

$$P=(\Delta+r)-(\bar\Delta+\bar r)-a \tag{13b}$$

with degeneracy $d(\Delta, r)d(\bar\Delta,\bar r)$. The value of a is zero for $\mathscr{E}_0^{(0)}(P)$ and $\mathscr{E}_1^{(0)}(P)$ and it is $\frac{1}{3}$ for $\mathscr{E}_1^{(1)}(P)$.

We now determine the irreps $(\Delta, \bar{\Delta})$ of the Virasoro algebra which contribute to $\mathscr{E}_0^{(0)}(P)$, $\mathscr{E}_1^{(0)}(P)$ and $\mathscr{E}_1^{(1)}(P)$. One obvious condition on the possible values of Δ and $\bar{\Delta}$ is that the momenta P are integer numbers (see (13*b*)).

In table 1 we show that the irreps $(0,0)$, $(\frac{2}{5},\frac{2}{5})$, $(\frac{7}{5},\frac{2}{5})$, $(\frac{2}{5},\frac{7}{5})$, $(\frac{7}{5},\frac{7}{5})$, $(3,0)$, $(0,3)$ give a perfect description of the 'experimental' levels. We omit some levels at $\mathscr{E}_0^{(0)}(2)=4.8$ and $\mathscr{E}_0^{(0)}(3)=3.8$ because we have not been able to compute numerically very high levels. $\mathscr{E}_0^{(0)}$ also probably contains the irreducible representation $(3,3)$ which would show up at $\mathscr{E}_0^{(0)}(0)=6$ but this goes beyond our computational abilities. It is important to notice that the irreps mentioned above form a closed subalgebra in the sense of short distance expansions (Belavin *et al* 1984).

In table 2 we show that the irreps $(\frac{1}{15},\frac{1}{15})$ and $(\frac{2}{3},\frac{2}{3})$ describe the 'observed' spectrum of $\mathscr{E}_1^{(0)}(P)$. In table 3 we observe that the irreps $(\frac{2}{5},\frac{1}{15})$, $(0,\frac{2}{3})$, $(\frac{7}{5},\frac{1}{15})$, $(3,\frac{2}{3})$ give the spectrum of $\mathscr{E}_1^{(1)}(P)$. Since the spectrum of $\mathscr{E}_1^{(1)}(P)$ is the same as that of $\mathscr{E}_1^{(2)}(P)$ (see (10)) but in this case a in (13*b*) is $\frac{2}{3}$ and the irreps for $\mathscr{E}_1^{(2)}(P)$ are obviously $(\frac{1}{15},\frac{2}{5})$, $(\frac{2}{3},0)$, $(\frac{1}{15},\frac{7}{5})$ and $(\frac{2}{3},3)$.

In conclusion, the following 18 irreps of the Virasoro algebra with $c=\frac{4}{5}$ describe the spectra of the three-state Potts model:

$$\mathscr{E}_0^{(0)}(P)\colon (0,0), (\tfrac{2}{5},\tfrac{2}{5}), (\tfrac{7}{5},\tfrac{2}{5}), (\tfrac{2}{5},\tfrac{7}{5}), (\tfrac{7}{5},\tfrac{7}{5}), (3,0), (0,3), (?)(3,3) \tag{14a}$$

$$\mathscr{E}_1^{(0)}(P)\colon (\tfrac{1}{15},\tfrac{1}{15}), (\tfrac{2}{3},\tfrac{2}{3}) \tag{14b}$$

$$\mathscr{E}_1^{(1)}(P)\colon (\tfrac{2}{5},\tfrac{1}{15}), (0,\tfrac{2}{3}), (\tfrac{7}{5},\tfrac{1}{15}), (3,\tfrac{2}{3}) \tag{14c}$$

$$\mathscr{E}_1^{(2)}(P)\colon (\tfrac{1}{15},\tfrac{2}{5}), (\tfrac{2}{3},0), (\tfrac{1}{15},\tfrac{7}{5}), (\tfrac{2}{3},3). \tag{14d}$$

We notice that the content of irreps of $\mathscr{E}_0^{(0)}(P)$ and $\mathscr{E}_1^{(0)}(P)$ coincides precisely with a prediction of Cardy (1986). In addition, through our use of twisted boundary conditions, we also find the para-fermionic representations (which have non-integer spin) listed in (14*c*) and (14*d*).

As the reader might have noticed, we have not considered the spectrum of the Hamiltonian with free boundary conditions. In this case instead of two Virasoro algebras, only one appears. The content of irreps for this case will be published elsewhere.

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